Minimax estimation of the scale parameter of the Gamma distribution (where the shape parameter is assumed to be known) for Quadratic and MLINEX loss functions

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ABSTRACT

This paper concerned with finding the minimax estimators of the scale parameter θ of Gamma distribution where the shape parameter α is known (letting $\alpha = 2$), under quadratic and Modified LINEX loss functions using Lehmann theorem (1950) . We take a numerical example by using Monte-Carlo simulation method, then we computed the estimated values and MSEs (with different sample sizes) for estimators and compared among them (MLE , MQL, MMLINEX) . The results are arranged and organized in tables and figures .

1. Introduction

The minimax estimation is non-classical approach in the estimation areas of statistical inference, which was introduced by Abraham Wald (1950) from the concept of Game theory . Von Neumann (1944) introduced the word minimax in game theory which is the optimum strategy of the second player in the two person zero game . According to Abraham Wald " minimax approach tries to guard against the worst by requiring that the chosen decision rule should provide maximum protection against the highest possible risk " . An estimator having this property is called a Minimax estimator (See Makhdoom (2011)).

Podder et al. (2004) studied the minimax estimator of the parameter of the Pareto distribution under Quadratic and MLINEX loss functions . In this paper we shall estimate the scale parameter of Gamma distribution by using technique of

minimax approach under quadratic and MLINEX loss function which are essentially in Bayesian approach . The basic principle of this approach is to minimize the loss of the statistician. The most important element in minimax approach is the specification of the prior distribution. In addition to the prior distribution, the minimax estimator for a particular model depends strongly on the loss function assumed .
The basic difference between the philosophy of the minimax and classical estimation is that minimax estimation considers the parameter of the distribution as a random variable, where in classical estimation regards it as a fixed point (See Dey (2008)) .

The Gamma distribution is the most widely used in life experiments, it has probability density function (p.d.f) with two parameters α and θ is :

$$f(X,\alpha,\theta)=\frac{X^{\alpha-1}}{\theta^{\alpha}\Gamma\alpha}e^{\frac{-X}{\theta}} \qquad X\geq 0 \quad , \qquad \alpha,\theta\geq 0$$

In this paper we will use the gamma distribution with one parameter θ where α is known (by letting $\alpha=2$), so it has probability density function (p.d.f) as follows :

$$f(X,\theta)=\frac{X}{\theta^{2}}e^{\frac{-X}{\theta}} \qquad X\geq 0 \quad , \quad \theta\geq 0 \qquad \ldots(1)$$

In this paper we shall derive the minimax estimator of the scale parameter θ of Gamma distribution under quadratic and MLINEX loss functions depends on Lehmann's Theorem (Lehmann, 1950) and can be stated as follows :

Lehmann 's Theorem :

Let $\tau=\{F_{\theta};\theta\in\Theta\}$ be a family of distribution functions and D a class of estimators of θ . Suppose that $d^{*}\in D$ is a Bayes estimator against a prior distribution $\xi^{*}(\theta)$ on the parameter space Θ and the risk function $R(d^{*},\theta)$ = constant on Θ ; then d^{*} is a minimax estimator of θ .

2. MAIN RESULTS

Theorem 2.1 : Let $X=(X_1, X_2,.....,X_n)$ be 'n' independently and identically distributed random variables drawn from the density (1) . Then $\hat{\theta}_{MQL} = \frac{\sum X_i}{2n+c}$ is the minimax estimator of the parameter θ for the quadratic loss function (QLF) for the type :

$$L(\theta, d_1) = (\frac{\theta - d_1}{\theta})^2 \qquad \ldots(2)$$

Where d_1 is the estimate of θ .

To prove Theorem 2.1 , it will be sufficient to show that $d_1 = \frac{\sum X_i}{2n+c}$ is a minimax estimator of θ for the loss function (2) .

First we have to find the Bayes' estimator d_1 of θ . If we can show that the risk function of d_1 is constant then Theorem 2.1 will be followed .

Let us assume that θ has non-informative prior density defined as :

$$g(\theta) \propto \frac{1}{\theta^c} \qquad \theta > 0 \quad , \quad c > 0 \qquad \ldots(3)$$

As pointed out in (1), the likelihood function of the distribution of $f(X,\theta)$ is given by :

$$L(x_1, x_2,.....,x_n|\theta) = \theta^{-2n}.\prod_{i=1}^{n} X_i.e^{\frac{-\sum X_i}{\theta}} \qquad \ldots(4)$$

$$\ln L(x_1, x_2,.....,x_n|\theta) = -2n\ln\theta + \text{constant} - \frac{\sum X_i}{\theta}$$

$$\frac{\partial \ln L(x_1, x_2,.....,x_n|\theta)}{\partial\theta} = \frac{-2n}{\theta} + \frac{\sum X_i}{\theta^2} = 0$$

Assuming that θ is unknown, then the maximum likelihood function (MLE) of the parameter θ can be estimated as follow :

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{2n}$$

Combining the likelihood function (4) and the prior $g(\theta)$ in (3), then the posterior distribution of θ given the random sample $X=(X_1, X_2,.....,X_n)$ is

$$\pi(\theta|X) = \frac{L(x_1, x_2, \ldots\ldots, x_n|\theta).g(\theta)}{\int_0^\infty L(x_1, x_2, \ldots\ldots, x_n|\theta).g(\theta)d\theta} = \frac{\prod_{i=1}^n X_i.\theta^{-2n-c} e^{\frac{-\sum X_i}{\theta}}}{\prod_{i=1}^n X_i \int_0^\infty \theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}} d\theta} = \frac{\prod_{i=1}^n X_i.\theta^{-2n-c} e^{\frac{-\sum X_i}{\theta}}}{\prod_{i=1}^n X_i.\frac{\Gamma 2n+c-1}{(\sum X_i)^{2n+c-1}}}$$

$$\therefore \pi(\theta|X) = \frac{(\sum X_i)^{2n+c-1} \theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1} \qquad \ldots(5)$$

Now for the QLF (2), the Bayes' estimator of θ is given by :

$$d_1 = \frac{E(\frac{1}{\theta})}{E(\frac{1}{\theta^2})}$$

$$\Rightarrow E(\frac{1}{\theta}) = \int_0^\infty \frac{1}{\theta}.\frac{(\sum X_i)^{2n+c-1}\theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1} d\theta = \frac{(\sum X_i)^{2n+c-1}}{\Gamma 2n+c-1} \int_0^\infty \theta^{-2n-c-1}.e^{\frac{-\sum X_i}{\theta}} d\theta$$

$$= \frac{(\sum X_i)^{2n+c-1}}{\Gamma 2n+c-1}.\frac{\Gamma 2n+c}{(\sum X_i)^{2n+c}} = \frac{2n+c-1}{\sum X_i}$$

And

$$E(\frac{1}{\theta^2}) = \int_0^\infty \frac{1}{\theta^2}.\frac{(\sum X_i)^{2n+c-1}\theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1} d\theta = \frac{(\sum X_i)^{2n+c-1}}{\Gamma 2n+c-1} \int_0^\infty \theta^{-2n-c-2}.e^{\frac{-\sum X_i}{\theta}} d\theta$$

$$= \frac{(\sum X_i)^{2n+c-1}}{\Gamma 2n+c-1}.\frac{\Gamma 2n+c+1}{(\sum X_i)^{2n+c+1}} = \frac{(2n+c)(2n+c-1)}{(\sum X_i)^2}$$

$$\therefore d_1 = \frac{\frac{2n+c-1}{\sum X_i}}{\frac{(2n+c)(2n+c-1)}{(\sum X_i)^2}} = \frac{\sum X_i}{2n+c} \qquad \ldots(6)$$

The risk function of the estimator d_1 is :

$$R(\theta) = EL(\theta, d_1) = E[\frac{1}{\theta^2}(\theta^2 - 2\theta d_1 + d_1^2]$$

$$= 1 - \frac{2}{\theta} Ed_1 + \frac{1}{\theta^2} Ed_1^2$$

$$Ed_1 = E(\frac{\sum X_i}{2n+c}) = \frac{\sum EX_i}{2n+c} = \frac{2n\theta}{2n+c}$$

And

$$Ed_1^2 = E(\frac{\sum X_i}{2n+c})^2 = \frac{6n^2\theta^2}{(2n+c)^2}$$

$$\Rightarrow R(\theta) = EL(\theta, d_1) = 1 - \frac{2}{\theta} \cdot \frac{2n\theta}{2n+c} + \frac{1}{\theta^2} \cdot \frac{6n^2\theta^2}{(2n+c)^2}$$

$$\therefore R(\theta) = 1 - \frac{4n}{2n+c} + \frac{6n^2}{(2n+c)^2} \qquad \ldots(7)$$

Which is a constant and independent on θ .

So, according to Lehmann's Theorem, it follows that

$\therefore d_1 = \frac{\sum X_i}{2n+c}$ is Minimax estimator of the parameter θ of Gamma distribution under Quadratic Loss Function $[L(\theta, d) = (\frac{\theta - d}{\theta})^2]$.

Theorem 2.2 : Let $X = (X_1, X_2, \ldots, X_n)$ be 'n' independently and identically distributed random variables drawn from the density (1) . Then $\hat{\theta}_{MML} = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{1}{c}} \sum X_i$ is the minimax estimator of the parameter θ for the MLINEX loss function for the type :

$$L(\theta, d_2) = w[(\frac{d_2}{\theta}) - c\ln(\frac{d_2}{\theta}) - 1] \qquad w > 0, c \neq 0 \qquad \ldots(8)$$

Where d_2 is the estimate of θ, w and c are two known parameters of the loss function .

In order to prove Theorem 2.2 , it will be sufficient to show that $\hat{\theta}_{MML} = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{1}{c}} \sum X_i$ is a minimax estimator of θ for the loss function (8) . First we have to find the Bayes' estimator d_2 of θ, if we can show that the risk function of d_2 is constant then Theorem 2.2 will be followed .

As before, by using the non-informative prior density, we get the posterior distribution of θ as :

$$\pi(\theta|X) = \frac{(\sum X_i)^{2n+c-1}\theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1}$$

Now for the MLINEX loss function (8), the derivation for the Bayes' estimator of θ is given by :

$$L(\theta, d_2) = w\frac{d_2^c}{\theta^c} - wc(\ln d_2 - \ln\theta) - w$$

$$= w\frac{d_2^c}{\theta^c} - wc\ln d_2 - wc\ln\theta - w$$

$$E[L(\theta, d_2) = wd_2^c.E\theta^{-c} - wc\ln d_2 + wc.E\ln\theta - w$$

$$\frac{\partial E[L(\theta, d_2)]}{\partial d} = wcd_2^{c-1}.E\theta^{-c} - \frac{wc}{d_2} = 0$$

$$wcd_2^{c-1}.E\theta^{-c} = \frac{wc}{d_2}$$

$$\therefore d_2^c = \frac{1}{E\theta^{-c}}$$

$$d_2 = (\frac{1}{E\theta^{-c}})^{\frac{1}{c}} = (E\theta^{-c})^{\frac{-1}{c}}$$

$$\Rightarrow \hat{\theta}_{BML} = (E\theta^{-c})^{\frac{-1}{c}}$$

Where $(E\theta^{-c})$ exists and finite and is given by :

$$E\theta^{-c} = \int_0^\infty \theta^{-c}\pi(\theta|X)d\theta$$

$$= \int_0^\infty \theta^{-c}.\frac{(\sum X_i)^{2n+c-1}\theta^{-2n-c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1}d\theta$$

$$= \int_0^\infty \frac{(\sum X_i)^{2n+c-1}\theta^{-2n-2c}.e^{\frac{-\sum X_i}{\theta}}}{\Gamma 2n+c-1}d\theta = \frac{(\sum X_i)^{2n+c-1}}{\Gamma 2n+c-1}.\frac{\Gamma 2n+2c-1}{(\sum X_i)^{2n+2c-1}}$$

$$\therefore d_2 = \hat{\theta}_{BML} = (\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{1}{c}}.\sum_{i=1}^{n} X_i$$

Note that when $c = 1$, then $d_2 = \frac{\Gamma 2n}{\Gamma 2n+1}\sum X_i = \frac{\sum X_i}{2n} = \hat{\theta}_{MLE}$.

The risk function of the estimator d_2.

$$R(\theta) = E[L(\theta, d_2) = wE[(\frac{d_2^c}{\theta^c}) - c\ln(\frac{d_2}{\theta}) - 1]$$

$$= w.[\frac{1}{\theta^c} Ed_2^c - cE\ln d_2 + c\ln\theta - 1]$$

$$\because d_2 = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{1}{c}}.\sum_{i=1}^{n} X_i \qquad \Rightarrow d_2^c = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}).(\sum_{i=1}^{n} X_i)^c$$

Let $\quad y = \sum_{i=1}^{n} X_i$

$$Ed_2^c = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}).Ey^c$$

$$Ey^c = \int y^c.\frac{1}{\theta^{2n}\Gamma 2n} y^{2n-1} e^{\frac{-y}{\theta}} dy = \frac{1}{\theta^{2n}\Gamma 2n}.\theta^{2n+c}\Gamma 2n + c$$

$$\therefore Ey^c = \frac{\Gamma 2n + c}{\Gamma 2n}.\theta^c$$

Then the first part of the risk function is :

$$\frac{1}{\theta^c} Ed_2^c = \frac{1}{\theta^c}(\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}).(\frac{\Gamma 2n + c}{\Gamma 2n}).\theta^c = \frac{\Gamma 2n + c}{\Gamma 2n}.\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}$$

And the second part of the risk function is :

$$cE\ln(d_2) = cE\ln[(\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{1}{c}}.y]$$

$$= cE\frac{1}{c}\ln(\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}) + cE\ln y = \ln(\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1}) + cE\ln y$$

$$E\ln y = \int_0^\infty \ln y.\frac{y^{2n-1}}{\theta^{2n}\Gamma 2n} e^{\frac{-y}{\theta}} dy$$

Let $\frac{y}{\theta} = t \quad \Rightarrow y = \theta t \qquad dy = \theta dt$

$$\int_0^\infty \ln(\theta t).\frac{(\theta t)^{2n-1}}{\theta^{2n}\Gamma 2n} e^{-t}.\theta dt = \frac{1}{\Gamma 2n}\int_0^\infty (\ln\theta + \ln t)t^{2n-1}e^{-t}dt$$

$$\frac{(\ln\theta)}{\Gamma 2n}\int_0^\infty t^{2n-1}e^{-t}dt + \frac{1}{\Gamma 2n}\int_0^\infty \ln t.t^{2n-1}e^{-t}dt$$

$$= \frac{\ln\theta}{\Gamma 2n}.\Gamma 2n + \frac{\Gamma' 2n}{\Gamma 2n}$$

Where $\Gamma' 2n = \int_0^\infty \ln t.t^{2n-1}e^{-t}dt$ is the first derivative of Γn with respect to n .

$$\therefore E\ln y = \ln\theta + \frac{\Gamma'2n}{\Gamma 2n}$$

$$\Rightarrow cE\ln(d_2) = \ln(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1}) + c(\ln\theta + \frac{\Gamma'2n}{\Gamma 2n})$$

$$\therefore cE\ln(d_2) = \ln(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1}) + c\ln\theta + c\frac{\Gamma'2n}{\Gamma 2n}$$

Now the risk function becomes :

$$R(\theta) = w[\frac{\Gamma 2n+c}{\Gamma 2n}.\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1} - \ln(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1}) - c\ln\theta - c\frac{\Gamma'2n}{\Gamma 2n} + c\ln\theta - 1]$$

$$\therefore R(\theta) = w[\frac{\Gamma 2n+c}{\Gamma 2n}.\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1} - \ln(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1}) - c\frac{\Gamma'2n}{\Gamma 2n} - 1]$$

Which is a constant, c and n are known and independent on θ .

Therefore $d_2 = \hat{\theta}_{MML} = (\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{1}{c}}.\sum_{i=1}^{n} X_i$ is the minimax estimator of the scale parameter θ of the Gamma distribution under the MLINEX loss function defined in (8) .

According to Wald (1950) the above statistical problem is equivalent to two person –zero-sum-game between (player-ll) and (player-l) .

Expectation of the loss function $L(\theta, d_i); i = 1,2$ is the risk function $R(\theta, d_i) = E[L(\theta, d_i)]$ which is the gain of player-l . $R(\xi, d_i)$ is the value of $\int R(\theta, d_i) d\xi(\theta)$, where $\xi(\theta)$ is the prior density of θ .

The number $R(\xi^*, d_i^*); i = 1,2$ is known to be the value of the game and ξ^* and d_i^* are the corresponding optimum strategies of player-l and player-ll . In statistical terms ξ^* is the least favourable prior density of θ and d_i^* is a minimax estimator of θ. In fact, the value of the game is the loss of the player-ll .

It has been shown that,

(l) $d_1^* = \hat{\theta}_{MQL} = \frac{\sum X_i}{2n+c}$ is the optimum strategy of player-ll for the quadratic loss function (2) and the value of the game is $R_Q(\xi^*, d_1^*) = 1 - \frac{4n}{2n+c} + \frac{6n^2}{(2n+c)^2}$.

(ll) $d_2^* = \hat{\theta}_{MML} = (\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{1}{c}}.\sum_{i=1}^{n} X_i$ is the optimum strategy of player-ll for the MLINEX loss function (8) and the value of the game is

$$R_M(\xi^*, d_2^*) = w[\frac{\Gamma 2n+c}{\Gamma 2n}.\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1} - \ln(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1}) - c\frac{\Gamma' 2n}{\Gamma 2n} - 1] \quad ; \quad c \neq 0, w > 0.$$

In both the cases $\xi^* = g(\theta) \propto \frac{1}{\theta}; \theta > 0$, is the optimum strategy for player-l , see (Dey & Basumatary 2008) .

3- Numerical Example

In this section we computed mean squared errors (MSEs) to compare the different estimators of the scale parameter θ of Gamma distribution, they are obtained by the Maximum Likelihood and Minimax for Quadratic and MLINEX loss function .the MSE are defined by :

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = var(\hat{\theta}) + [Bias(\hat{\theta})]^2$$

The estimated values of the scale parameter θ and MSE of the estimators are computed by using Monte-Carlo Simulation method for Gamma distribution .

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} X_i}{2n} \quad , \quad d_1 = \hat{\theta}_{MQL} = \frac{\sum_{i=1}^{n} X_i}{2n+c}$$

$$d_2 = \hat{\theta}_{MMLINEX} = (\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{1}{c}}.\sum_{i=1}^{n} X_i$$

$$E(X) = 2\theta \qquad \Rightarrow E(\hat{\theta}_{MLE}) = \frac{E(\sum_{i=1}^{n} X_i)}{2n} = \theta$$

So $\hat{\theta}_{MLE}$ is unbiased estimator for θ .

Also

$$E(\hat{\theta}_{MQL}) = \frac{E(\sum_{i=1}^{n} X_i)}{2n+c} = \frac{2n\theta}{2n+c}$$

i.e. $\hat{\theta}_{MQL}$ is biased estimator for θ .

And

$$E(\hat{\theta}_{MMLINEX}) = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{1}{c}} . E(\sum_{i=1}^{n} X_i)$$

$$= 2K^{\frac{1}{c}} n\theta$$

Where $K = (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})$

i.e. $\hat{\theta}_{MMLINEX}$ is biased estimator for θ .

We have $V(X) = 2\theta^2$

Therefore

$$V(\hat{\theta}_{MLE}) = V(\frac{\sum_{i=1}^{n} X_i}{2n}) = \frac{\theta^2}{2n}$$

$$V(\hat{\theta}_{MQL}) = V(\frac{\sum_{i=1}^{n} X_i}{2n + c}) = \frac{2n\theta^2}{(2n + c)^2}$$

$$V(\hat{\theta}_{MMLINEX}) = 2n\theta^2 (\frac{\Gamma 2n + c - 1}{\Gamma 2n + 2c - 1})^{\frac{2}{c}}$$

To compute MSEs for estimators, as follows :

$$MSE(\hat{\theta}_{MLE}) = V(\hat{\theta}_{MLE}) + (Bias(\hat{\theta}))^2$$

Where $Bias(\hat{\theta}_{MLE}) = E\hat{\theta}_{MLE} - \theta = \theta - \theta = 0$

$$MSE(\hat{\theta}_{MLE}) = \frac{\theta^2}{2n}$$

And for $\hat{\theta}_{MQL}$, the biased part is :

$$Bias(\hat{\theta}_{MQL}) = E\hat{\theta}_{MQL} - \theta = \frac{-c\theta}{2n + c}$$

$$MSE(\hat{\theta}_{MQL}) = \frac{\theta^2 (2n + c^2)}{(2n + c)^2}$$

And for $\hat{\theta}_{MMLINEX}$, the biased part is :

$$Bias(\hat{\theta}_{MMLINEX}) = E\hat{\theta}_{MMLINEX} - \theta = \theta(2nK^{\frac{1}{c}} - 1)$$

$$MSE(\hat{\theta}_{MMLINEX}) = \theta^2(1 + 4nK^{\frac{2}{c}} - 4nK^{\frac{1}{c}})$$

Therefore the efficiency of $\hat{\theta}_{MQL}$ with respect to $\hat{\theta}_{MLE}$ of θ is :

$$Efficiency = \frac{\frac{\theta^2}{2n}}{\frac{2n\theta^2}{(2n+c)^2}} = \frac{(2n+c)^2}{4n^2} > 1$$

It means that minimax estimator under QLF is more efficient than MLE.

The efficiency of $\hat{\theta}_{MMLINEX}$ with respect to $\hat{\theta}_{MQL}$ of θ is :

$$Efficiency = \frac{\frac{2n\theta^2}{(2n+c)^2}}{2n\theta^2(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{2}{c}}} = \frac{1}{K^{\frac{2}{c}}(2n+c)^2} < 1$$

It means that minimax estimator under QLF is more efficient than MMLINEX.

Also, the efficiency of $\hat{\theta}_{MMLINEX}$ with respect to $\hat{\theta}_{MLE}$ of θ is :

$$Efficiency = \frac{\frac{\theta^2}{2n}}{2n\theta^2(\frac{\Gamma 2n+c-1}{\Gamma 2n+2c-1})^{\frac{2}{c}}} = \frac{1}{4n^2K^{\frac{2}{c}}} < 1$$

It means that minimax estimator under MLE is more efficient than MMLINEX .

Table 1: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = 1$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.108589** **0.1**	**1.007809** **0.090909**	**1.108589** **0.1**
10	Estimated values MSE	**1.311877** **0.05**	**1.249407** **0.047619**	**1.311877** **0.05**
15	Estimated values	**1.229531** **0.033333**	**1.189868** **0.032258**	**1.229531** **0.033333**

	MSE			
20	Estimated values MSE	**1.139013** **0.025**	**1.111232** **0.02439**	**1.139013** **0.025**
25	Estimated values MSE	**0.766389** **0.02**	**0.751362** **0.019608**	**0.766389** **0.02**
30	Estimated values MSE	**1.060251** **0.016667**	**1.04287** **0.016393**	**1.060251** **0.016667**
40	Estimated values MSE	**1.058478** **0.0125**	**1.045411** **0.012346**	**1.058478** **0.0125**
50	Estimated values MSE	**0.919146** **0.01**	**0.910046** **0.009901**	**0.919146** **0.01**

Table 2: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = 1.5$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**0.964224** **0.1**	**0.838456** **0.092628**	**0.897357** **0.09142**
10	Estimated values MSE	**0.971529** **0.05**	**0.903748** **0.048134**	**0.936527** **0.047760**
15	Estimated values MSE	**0.911702** **0.033333**	**0.868288** **0.032501**	**0.889514** **0.032322**
20	Estimated values MSE	**1.164972** **0.025**	**1.122865** **0.024531**	**1.143566** **0.024427**
25	Estimated values MSE	**0.702579** **0.02**	**0.682116** **0.019700**	**0.692204** **0.019631**
30	Estimated values MSE	**0.760290** **0.016666**	**0.741746** **0.016458**	**0.750914** **0.016410**
40	Estimated values MSE	**0.956619** **0.0125**	**0.939012** **0.012382**	**0.947740** **0.012355**
50	Estimated values MSE	**1.052693** **0.01**	**1.037136** **0.009925**	**1.044860** **0.009907**

Table 3: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = 2$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$

5	Estimated values MSE	**1.355464** **0.1**	**1.129553** **0.097222**	**1.17978** **0.092557**
10	Estimated values MSE	**0.972396** **0.05**	**0.883997** **0.049587**	**0.904799** **0.048122**
15	Estimated values MSE	**0.951498** **0.033333**	**0.89203** **0.033203**	**0.906303** **0.032498**
20	Estimated values MSE	**1.056068** **0.025**	**1.005779** **0.024943**	**1.017971** **0.02453**
25	Estimated values MSE	**1.09872** **0.02**	**1.056462** **0.01997**	**1.066769** **0.019699**
30	Estimated values MSE	**0.869327** **0.016667**	**0.841284** **0.016649**	**0.848152** **0.016458**
40	Estimated values MSE	**1.120416** **0.0125**	**1.093089** **0.012493**	**1.099816** **0.012383**
50	Estimated values MSE	**1.035019** **0.01**	**1.014724** **0.009996**	**1.019735** **0.009925**

Table 4: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = 3$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.335213** **0.1**	**1.027087** **0.112426**	**1.029121** **0.11196**
10	Estimated values MSE	**0.954568** **0.05**	**0.830059** **0.05482**	**0.830582** **0.054725**
15	Estimated values MSE	**1.015064** **0.033333**	**0.922785** **0.035813**	**0.923068** **0.035779**
20	Estimated values MSE	**1.015348** **0.025**	**0.94451** **0.026501**	**0.94468** **0.026485**
25	Estimated values MSE	**1.081713** **0.02**	**1.020484** **0.021004**	**1.020606** **0.020995**
30	Estimated values MSE	**1.123377** **0.016667**	**1.069883** **0.017385**	**1.069973** **0.01738**
40	Estimated values MSE	**0.865288** **0.0125**	**0.834012** **0.012919**	**0.834053** **0.012917**
50	Estimated values MSE	**0.972007** **0.01**	**0.943696** **0.010274**	**0.943726** **0.010273**

Table 5: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = -1$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.047747** **0.1**	**1.164163** **0.135802**	**1.496781** **0.387755**
10	Estimated values MSE	**1.372332** **0.05**	**1.44456** **0.058172**	**1.614508** **0.100346**
15	Estimated values MSE	**1.004659** **0.033333**	**1.039302** **0.036861**	**1.116288** **0.053498**
20	Estimated values MSE	**1.03128** **0.025**	**1.057723** **0.026956**	**1.114897** **0.035793**
25	Estimated values MSE	**0.867788** **0.02**	**0.885498** **0.021241**	**0.923178** **0.026709**
30	Estimated values MSE	**0.935651** **0.016667**	**0.951509** **0.017524**	**0.984895** **0.021237**
40	Estimated values MSE	**1.022024** **0.0125**	**1.034961** **0.012979**	**1.061843** **0.015011**
50	Estimated values MSE	**0.969582** **0.01**	**0.979375** **0.010305**	**0.999569** **0.011585**

Table 6: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = -1.5$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.023746** **0.1**	**1.204408** **0.169550**	**1.640180** **0.619248**
10	Estimated values MSE	**1.097747** **0.05**	**1.186753** **0.065011**	**1.35134** **0.129136**
15	Estimated values MSE	**0.943564** **0.033333**	**0.993226** **0.039705**	**1.078550** **0.064018**
20	Estimated values MSE	**1.234241** **0.025**	**1.282328** **0.028504**	**1.361970** **0.041153**
25	Estimated values MSE	**1.069395** **0.02**	**1.102469** **0.022213**	**1.156130** **0.029954**

30	Estimated values MSE	**1.293971** **0.016666**	**1.333990** **0.018190**	**1.150210** **0.023410**
40	Estimated values MSE	**0.962113** **0.0125**	**0.980497** **0.013347**	**1.009440** **0.016180**
50	Estimated values MSE	**1.050631** **0.01**	**1.066631** **0.010539**	**1.091570** **0.012313**

Table 7: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = -2$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.198018** **0.1**	**1.497523** **0.21875**	**2.187272** **1.015183**
10	Estimated values MSE	**1.361468** **0.05**	**1.512742** **0.074074**	**1.757648** **0.168011**
15	Estimated values MSE	**1.236385** **0.033333**	**1.324698** **0.043367**	**1.454851** **0.077376**
20	Estimated values MSE	**0.947319** **0.025**	**0.997178** **0.030471**	**1.067508** **0.047843**
25	Estimated values MSE	**0.815537** **0.02**	**0.849518** **0.023438**	**0.896249** **0.033949**
30	Estimated values MSE	**0.956076** **0.016667**	**0.989045** **0.019025**	**1.033638** **0.026062**
40	Estimated values MSE	**0.962989** **0.0125**	**0.987681** **0.013807**	**1.020408** **0.01759**
50	Estimated values MSE	**1.050631** **0.01**	**1.072073** **0.010829**	**1.100152** **0.013187**

Table 8: Estimated values and MSEs of different estimators for the parameter of θ of Gamma distribution when $\theta = 1$ and $c = -3$.

Sample Size	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{MQL}$	$\hat{\theta}_{MMLINEX}$
5	Estimated values MSE	**1.267952** **0.1**	**1.811361** **0.387755**	**3.238813** **3.068528**
10	Estimated values MSE	**1.323853** **0.05**	**1.557474** **0.100346**	**1.894446** **0.288158**

15	Estimated values MSE	**1.319738** **0.033333**	**1.466375** **0.053498**	**1.650628** **0.115006**
20	Estimated values MSE	**1.18126** **0.025**	**1.277038** **0.035793**	**1.390119** **0.065884**
25	Estimated values MSE	**1.059756** **0.02**	**1.1274** **0.026709**	**1.204476** **0.044484**
30	Estimated values MSE	**0.987412** **0.016667**	**1.039381** **0.021237**	**1.09725** **0.032955**
40	Estimated values MSE	**0.987299** **0.0125**	**1.025765** **0.015011**	**1.067415** **0.021196**
50	Estimated values MSE	**0.905701** **0.01**	**0.933713** **0.011585**	**0.963548** **0.015398**

Fig 1: Graph of MSE for differnet values of n under MLE , MQL and MMLINEX when Theta=1 and c=1

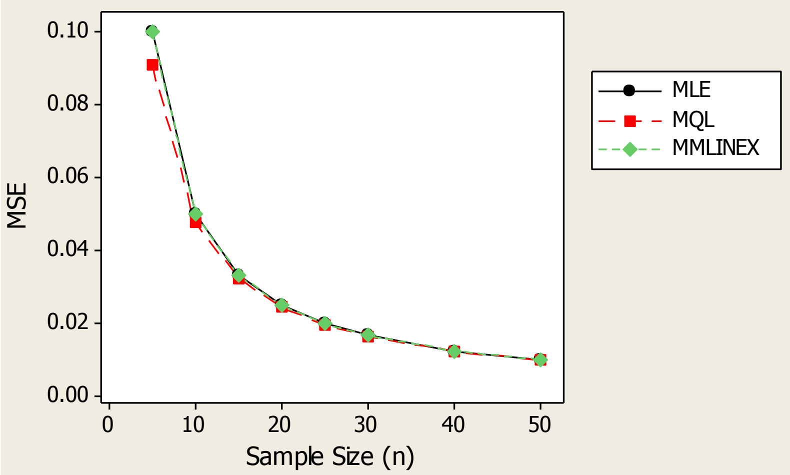

Fig 2: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=1.5

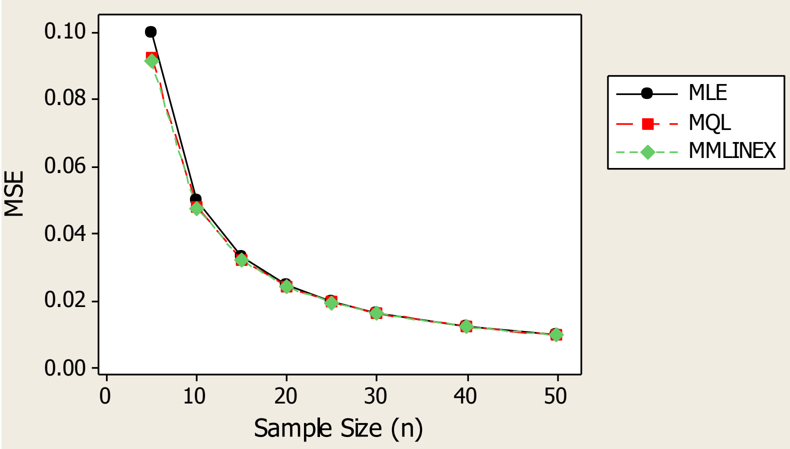

Fig 3: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=2

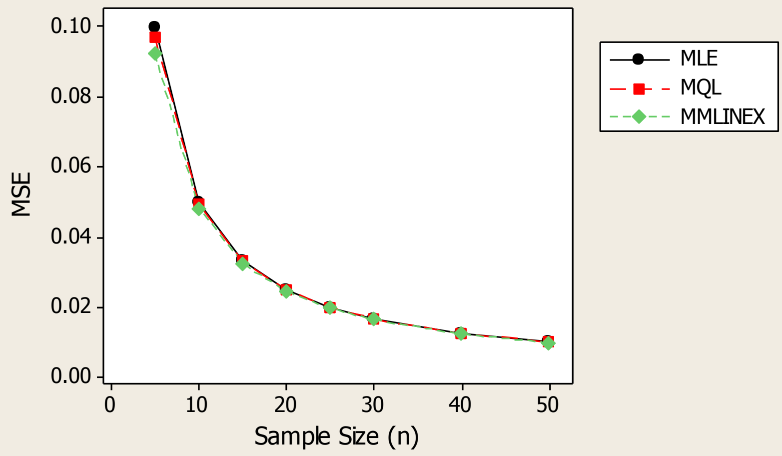

Fig 4: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=3

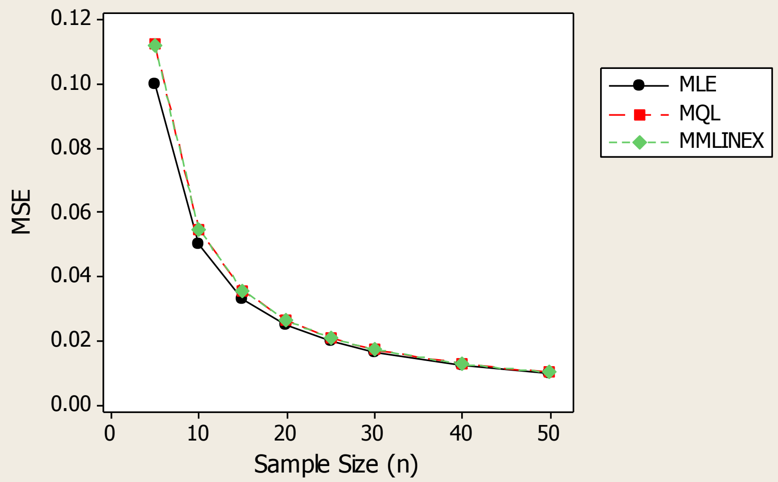

Fig 5: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=-1

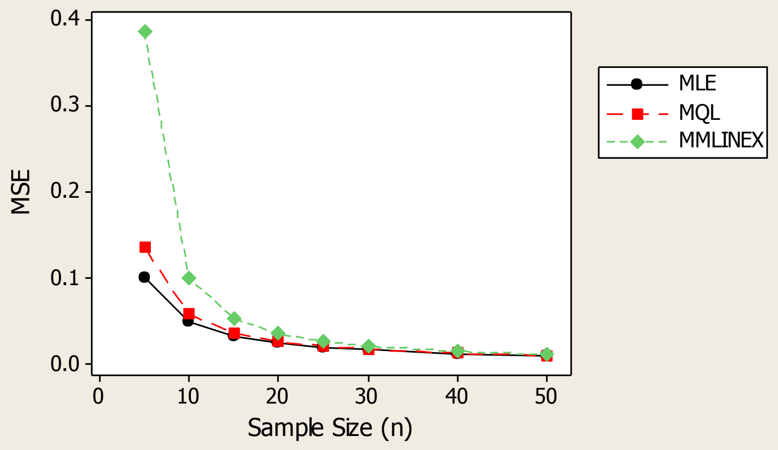

Fig 6: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=-1.5

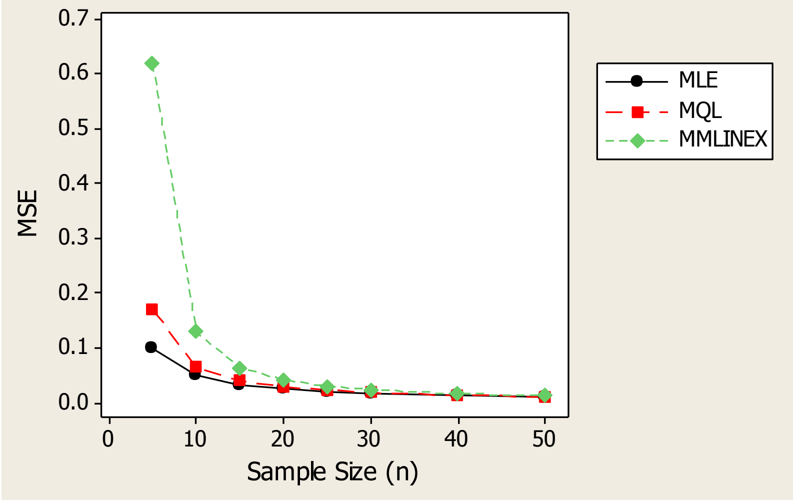

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Fig 7: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=-2

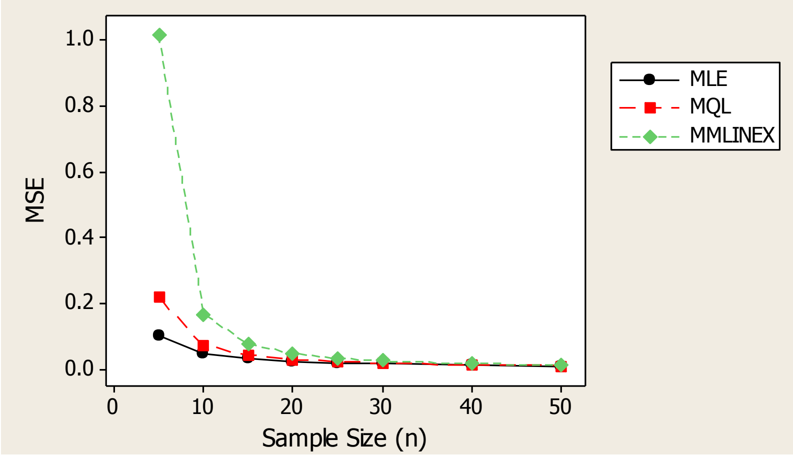
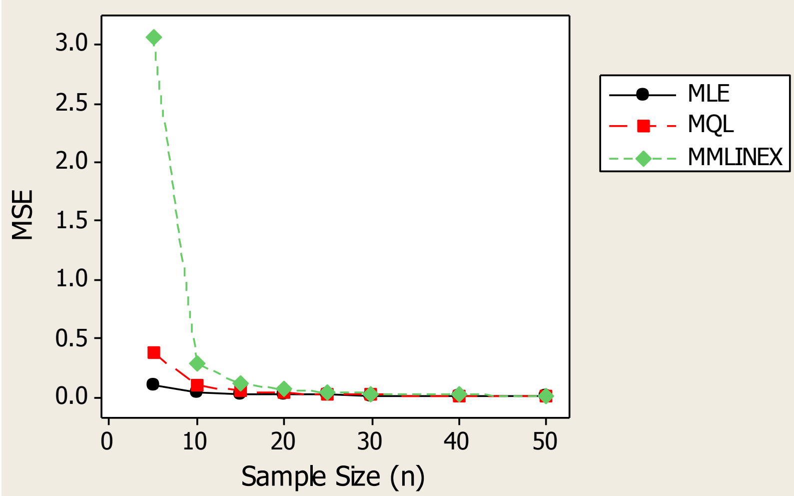


Fig 8: Graph of MSE for different values of under MLE , MQL and MMLINEX when Theta=1 and c=-3



4- Conclusion

By using Monte-Carlo simulation method for the parameter θ of the gamma distribution, we can see :

1- From table 1 and figures 1 (if $0 < c < 1$) that the minimax estimator under quadratic loss function is better than maximum likelihood estimator and the minimax estimator under modified linex loss function, we can see also that $MSE(\hat{\theta}_{MQL})$ and $MSE(\hat{\theta}_{MMLINEX})$ are the same .

2- From tables 2 and 3 , figures 1 and 2 (if $1 \le c \le 2$) that the minimax estimator under modified linex loss function is better than maximum likelihood estimator and the minimax estimator under quadratic loss function .

3- From tables 4 ,5,6,7 and 8, figures 4,5,6,7 and 8 (if $c \ge 3$ and $c < 0$) that the maximum likelihood estimator is better than the minimax estimator under quadratic loss function and the minimax estimator under modified linex loss function .

4- The MSEs are decrease when the sample sizes have been increased .

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