

## **E – Bayesian and Hierarchical Estimation of Maxwell – Boltzman Distribution**

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### **Abstract**

This paper is devoted to compare the E-Bayesian and hierarchical Bayesian estimation of the shape parameter corresponding to the Burr Type *II* distribution. The E-Bayesian and hierarchical Bayesian estimation are obtained under the squared error loss function (SELF), LINEX loss function, general entropy loss function (GELF), and precautionary loss function (PLF). Properties of the E-Bayesian and hierarchical Bayesian estimation are investigated. Comparisons among all estimates are performed in terms of mean square error (MSE) via Monte Carlo simulation. Numerical computations showed that E-Bayesian estimates are more efficient than the hierarchical Bayesian estimates. Real data is used to represent these estimates.

**Key words:** E-Bayesian estimation, hierarchical Bayesian estimation, Burr Type II, loss functions, Monte Carlo simulation.

### **1. Introduction**

Burr distribution was first introduced in the literature by Burr (1942). Inference on the parameters of Burr distributions has been the subject of investigation by many authors including Evans and Ragab (1983) and Al – Marzoug and Ahmad (1985). The Burr X distribution plays a key role in analysis of life time, agriculture, health, biology, actuarial sciences and survival analysis. It is also known as the generalized Rayleigh distribution.

Han (2009) introduced E-Bayesian estimation method, to estimate failure rate. The method is suitable for the censored or truncated data with small sample sizes and high reliability. The definition and properties of E-Bayesian estimation are given. A real data set showed that the method is both efficient and easy to apply.

Feroze and Aslam (2012) considered Bayes estimation of the parameter of Burr type *XI* distribution. The posterior analysis is carried out under the assumption of eight priors (informative, non-informative, single and mixture of priors). The entropy and precautionary loss functions are used for estimation. A simulation study is conducted to discuss the applicability of the results. The study suggested that performance of the estimates under the mixture priors is better than those under single priors.

Azimi *et al.* (2013) introduced a progressively Type II censored sample from a generalized half logistic distribution, Bayesian and E-Bayesian estimators are obtained under LINEX and squared-error loss functions, for the parameter and reliability function. Monte Carlo simulation method is used to generate a progressive Type-II censored data from generalized half logistic distribution, then this data is used to compute the estimations of the parameter and compare both the methods used with different random schemes.

Okasha (2014) concerned with using E-Bayesian method for computing estimates of the unknown parameter, reliability and hazard functions of Lomax distribution based on Type-II censored data. These estimates are derived based on a conjugate prior for the parameter under the balanced squared error loss function. A comparison between the new method and the corresponding Bayes and maximum likelihood techniques is conducted using the Monte Carlo simulation.

Gonzalez-Lopez *et al.* (2015) introduced E-Bayesian approach on exponential distribution under squared error loss function. They used three prior distributions to investigate its impact on the E-Bayesian approach. They showed in real examples and also by simulations, how the procedure behaves. In the simulation study they explored the impact on this estimation approach, when the number of components of the system increases.

Reyad and Ahmed (2015) focused on Bayesian and E-Bayesian estimation for the unknown shape parameter of the Gumbel Type-II distribution based on type-II censored samples. These estimators are obtained under symmetric loss function [squared error loss (SELF)] and various asymmetric loss functions [LINEX loss function (LLF), Degroot loss function (DLF), Quadratic loss function (QLF) and minimum expected loss function (MELF)]. Comparisons between the E – Bayesian estimators with the associated Bayesian estimators are investigated through a simulation study.

Nasiri and Esfandyarifar (2016) introduced E-Bayesian estimation to estimate the parameter of logarithmic series distribution. In addition, E-Bayesian, Bayesian and maximum likelihood estimation are compared by MSE using a simulation.

Reyad *et al.* (2016) introduced a comparative study for the E-Bayesian criteria with three various Bayesian approaches; Bayesian, hierarchical Bayesian and empirical Bayesian. They concerned to estimate the shape parameter and the hazard function of the Gompertz distribution based on type-II censoring. All estimators are obtained under symmetric loss function [squared error loss (SELF)] and three asymmetric loss functions [quadratic loss function (QLF), entropy loss function (ELF) and LINEX loss function (LLF)]. Comparisons among all estimators are achieved in terms of mean square error (MSE) via Monte Carlo simulation.

Gupta and Gupta (2017) are concerned with using Bayesian and E-Bayesian method of estimation to find estimates for the shape parameter of Exponentiated Inverted Weibull

distribution. These estimators are derived by using different loss functions. Bayesian estimates are derived by using informative prior.

Han (2018) concerned E-Bayesian method for computing estimates of the Exponentiated distribution family parameter. Based on the LINEX loss function, formulas of E-Bayesian estimation for unknown parameter are given, these estimates are derived based on a conjugate prior. A comparison between the new method and the corresponding maximum likelihood techniques is conducted using the Monte Carlo simulation. Finally, combined with the golfer's income data practical problem are calculated, the results showed that the proposed method is feasible and convenient for application.

Reyad and Ahmed (2018) concerned with comparing E-Bayesian and Bayesian methods for estimating the shape parameters of two-component mixture of inverse Lomax distribution based on Type – I censored data. Based on the squared error loss (SELF), minimum expected loss (MELF), Degroot loss (DLF), precautionary loss (PLF), LINEX loss (LLF) and entropy loss (ELF) functions, formulas of E-Bayesian and Bayesian estimations are given. These estimates are derived based on a conjugate gamma prior and uniform hyper prior distributions. Comparisons among all estimates are performed in terms of absolute bias (*Abias*) and mean square error (MSE) via Monte Carlo simulation. Numerical computations showed that E-Bayesian estimates are more efficient than the corresponding Bayesian estimates.

This paper is divided into eight sections. The first section is the introduction. The second section contains the Bayesian Estimation for Burr Type II distribution. The posterior distribution is contained in Section 3. E – Bayesian estimation is contained in Section 4. The fifth section is devoted to the hierarchical Bayesian estimation. Section six contains properties of the E – Bayesian and hierarchical Bayesian estimates. Monte Carlo simulation is contained in Section 7. The application using real data is contained in Section 8.

## 2. Bayesian Estimation:

The probability density function (*pdf*) and cumulative distribution function (*cdf*) of the Burr Type II distribution are given respectively as:

$$f(x, \theta) = \theta e^{-x} (1 + e^{-x})^{-\theta-1}, -\infty < X < \infty; \theta > 0 \quad (1)$$

$$F(x, \theta) = (1 + e^{-x})^{-\theta}, -\infty < X < \infty; \theta > 0 \quad (2)$$

where  $\theta$  is the shape parameter.

### 2.1 Loss Functions and Posterior Distribution

In this section, we refer to the various loss functions considered in this paper and derive the posterior distribution.

We will use the following loss functions:

1. The squared error loss function (SELF) is a symmetric loss function and takes the form:

$$L(\theta, \hat{\theta}) = c(\theta - \hat{\theta})^2 \quad (3)$$

where  $c$  denotes a constant and  $\hat{\theta}$  is an estimator. The Bayes estimator with respect to a quadratic loss function is the mean of the posterior distribution which takes the form:

$$\hat{\theta}_B = E(\theta | \underline{X}) = \int_0^{\infty} \theta \pi(\theta | x) d\theta = \int_0^{\infty} \theta \pi(\theta | x) d\theta \quad (4)$$

2. Varian (1975) introduced the linear-exponential (LINEX) loss function as an asymmetric loss function defined as:

$$L(\theta, \hat{\theta}) = e^{v(\hat{\theta}-\theta)} - v(\hat{\theta} - \theta) - 1 \quad (5)$$

where  $\theta$  is a univariate parameter and  $v \neq 0$ . The constant  $v$  determines the shape of the loss function. If the constant  $v > 0$  and the error  $(\hat{\theta} - \theta)$  is positive the LINEX loss function is almost exponential and for negative errors almost linear, in these situation overestimation is a more serious problem than underestimation. If  $v < 0$  underestimation is more important than overestimation. For small values of  $|v|$  the loss is an almost symmetric and behaves like the SE loss function. Under LINEX loss function, the Bayes estimator  $\hat{\theta}$  of  $\theta$  is given by:

$$\hat{\theta} = \frac{-1}{v} \ln E_{\theta}(e^{-v\theta} | \underline{X}) = \frac{-1}{v} \ln \left( \frac{(b+T)^{n+a}}{\Gamma(n+a)} \right) \int_0^{\infty} e^{-v\theta} \theta^{n+a-1} e^{-\theta(b+T)} d\theta \quad (6)$$

where  $E_{\theta}$  stands for positive expectation [See Zellner (1986)]

3. Despite the flexibility of the LINEX loss function for the estimation of a location parameter, it appears not to be suitable for the estimation of scale parameters and other quantities. For these reasons Basu and Ibrahim (1991) proposed the modified LINEX loss function. Calabria and Pulcini (1996) presented another alternative to the modified LINEX loss function named general entropy (GE) loss function when it appears to be realistic to express the loss in terms of the ratio  $\left(\frac{\hat{\theta}}{\theta}\right)$  and defined it as:

$$L(\theta, \hat{\theta}) = \left(\frac{\hat{\theta}}{\theta}\right)^q - q \log \left(\frac{\hat{\theta}}{\theta}\right) - 1 \quad (7)$$

whose minimum occurs at  $\hat{\theta} = \theta$ . This loss function is a generalization of the entropy loss used by several authors where the shape parameter  $q = 1$  [Dey and Liu (1992)]. When  $q > 0$  a positive error  $(\hat{\theta} > \theta)$  causes serious consequences than a negative error. The Bayes estimator  $\hat{\theta}_{GE}$  of  $\theta$  under GE loss is

$$\hat{\theta}_{GE} = [E_{\theta}(\theta^{-q})]^{-1/q} = \int_0^{\infty} \theta^{-1} \pi(\theta | x) d\theta \quad (8)$$

Provided that the expectation  $[E_{\theta}(\theta^{-q})]^{-1/q}$  exists and finite where  $E_{\theta}$  denotes the expected value with respect to the posterior function of  $\theta$

4. Precautionary loss function is very useful when underestimation may lead to serious consequences. A very useful and simple asymmetric loss function is:

$$L(\theta, \hat{\theta}) = \frac{(\theta - \hat{\theta})^2}{\hat{\theta}\theta} \quad (10)$$

The Bayes estimator under this asymmetric loss function can be obtained by solving the following equation:

$$(\hat{\theta}_B)^2 = \frac{E(\theta|\underline{X})}{E(\theta^{-1}|\underline{X})} \quad (11)$$

### 3. The posterior distribution

The likelihood function can be obtained as:

$$L(\theta|\underline{X}) = \theta^n e^{-\sum_{i=1}^n x_i} \prod_{i=1}^n (1 + e^{-x_i})^{-\theta-1} \quad (12)$$

Assuming  $\theta$  is known then the likelihood function in Eq. (12) become

$$L(\theta|\underline{X}) = \theta^n e^{-\theta \sum_{i=1}^n \ln(1+e^{-x_i})} = \theta^n e^{-\theta T} \quad (13)$$

where

$$\underline{X} = (x_1, x_2, \dots, x_n), T = \sum_{i=1}^n \ln(1 + e^{-x_i}) \quad (14)$$

We can use the gamma distribution as a conjugate prior distribution of  $\theta$  with parameters  $a$  and  $b$  as:

$$\pi(\theta|a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}, \theta > 0, a, b > 0 \quad (15)$$

where  $\Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt$  is Gamma function, hyper parameters  $a > 0, b > 0$ ,

The posterior distribution of  $\theta$  can be obtained by combining Eqs. (13) and (15) to be:

$$\pi(\theta|x) = L(\theta|\underline{X})\pi(\theta|a, b) = \frac{(b + T)^{a+n}}{\Gamma(a + n)} \theta^{n+a-1} e^{-\theta(b+T)}, \theta, a, b > 0 \quad (16)$$

#### 4. The E-Bayesian estimation

In this section, we have obtained the E-Bayesian estimate for shape parameter of Burr Type II under SELF, LINEX, GELF and PLF.

Let the prior distribution of  $\theta$  be its conjugated distribution – Gamma distribution with density function. According to Han (1997),  $a$  and  $b$  should be selected to guarantee that  $\pi(\theta|a, b)$  is a decreasing function of  $\theta$ . The derivative of  $\pi(\theta|a, b)$  with respect to  $\theta$

$$\frac{d\pi(\theta|a, b)}{d\theta} = \frac{b^a}{\Gamma(a)} [\theta^{a-1} e^{-b\theta} (-b) + e^{-b\theta} (a-1)\theta^{a-2}] \quad (17)$$

Since  $\theta > 0, a > 0$ , and  $b > 0$ , then  $0 < a < 1$  and  $b > 0$  due to  $\frac{d\pi(\theta|a, b)}{d\theta} < 0$ , that is,  $d\pi(\theta|a, b)$  is a decreasing function of  $\theta$ , given  $0 < a < 1$ . The larger  $b$  is the thinner the tail of the Gamma density function will be considering the robustness of Bayesian estimate. The thinner tailed prior distribution often reduces the robustness of Bayesian estimate. Accordingly,  $b$  should be larger than a given upper bound  $c$  where  $c > 0$  is a constant to be determined. Thereby, the hyper parameters  $a$  and  $b$  should be selected with the restriction of  $0 < a < 1$  and  $0 < b < c$ . When  $a = 1$ ,  $\pi(\theta|a, b)$  is a decreasing function of  $\theta$ . Accordingly,  $b$  should not be too big while  $a = 1$ . It is better to choose  $b$  below a given upper bound  $c$  ( $c$  is a positive constant). Thereby, the range of hyperparameter  $b$  may be considered as  $0 < b < c$ .

Then, we can use the following hyper prior distributions of  $a, b$  introduced by Han

$$\pi_1(a, b) = \frac{2(c-b)}{c^2}, 0 < b < c, 0 < a < 1 \quad (18)$$

$$\pi_2(a, b) = \frac{1}{c}, 0 < b < c, 0 < a < 1 \quad (19)$$

and

$$\pi_3(a, b) = \frac{2b}{c^2}, 0 < b < c, 0 < a < 1 \quad (20)$$

##### 4.1 The E-Bayesian estimation under SELF Loss Function

The Bayes estimator with respect to the quadratic loss function is the mean of the posterior distribution as follows:

$$E(\theta|x) = \int_0^\infty \theta \pi(\theta|x) d\theta = \int_0^\infty \theta \frac{(b+T)^{a+n}}{\Gamma(a+n)} \theta^{n+a-1} e^{-\theta(b+T)} d\theta = \frac{a+n}{b+T} \quad (21)$$

The E – Bayesian estimates  $\hat{\theta}_{Ese1}$ ,  $\hat{\theta}_{Ese2}$  and  $\hat{\theta}_{Ese3}$  based on  $\pi_1(a, b)$ ,  $\pi_2(a, b)$  and  $\pi_3(a, b)$  respectively relative to SELF are the following:

$$1. \hat{\theta}_{Ese1} = \int_0^1 \int_0^c \pi_1(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left(\frac{n+a}{b+T}\right) \frac{2(c-b)}{c^2} dbda \quad (22)$$

$$2. \hat{\theta}_{Ese2} = \int_0^1 \int_0^c \pi_2(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left(\frac{n+a}{b+T}\right) \frac{1}{c} dbda \quad (23)$$

and

$$3. \hat{\theta}_{Ese3} = \int_0^1 \int_0^c \pi_3(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left(\frac{n+a}{b+T}\right) \frac{2b}{c^2} dbda \quad (24)$$

Eqs. (22) – (24) can be solved numerically using Mathcad program.

#### 4.2 The E-Bayesian estimation under LINEX Loss Function

The Bayes estimator with respect to the LINEX loss function is the mean of the posterior distribution as follows:

$$\begin{aligned} \hat{\theta}_B &= \frac{-1}{v} \ln E(e^{-v\theta} | \underline{X}) \\ &= \frac{-1}{v} \ln \int_0^\infty e^{-v\theta} \frac{(b+T)^{a+n}}{\Gamma(a+n)} \theta^{n+a-1} e^{-\theta(b+T)} d\theta \\ &= \frac{-1}{v} \ln \left( \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \right) \end{aligned} \quad (25)$$

The E – Bayesian estimates  $\hat{\theta}_{Elnx1}$ ,  $\hat{\theta}_{Elnx2}$  and  $\hat{\theta}_{Elnx3}$  based on  $\pi_1(a, b)$ ,  $\pi_2(a, b)$  and  $\pi_3(a, b)$  respectively relative to LINEX are the following:

$$1. \hat{\theta}_{Elnx1} = \int_0^1 \int_0^c \pi_1(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{-1}{v} \ln \left( \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \right) \frac{2(c-b)}{c^2} dbda \quad (26)$$

$$2. \hat{\theta}_{Elnx2} = \int_0^1 \int_0^c \pi_2(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{-1}{v} \ln \left( \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \right) \frac{1}{c} dbda \quad (27)$$

and

$$3. \hat{\theta}_{Elnx3} = \int_0^1 \int_0^c \pi_3(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{-1}{v} \ln \left( \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \right) \frac{2b}{c^2} dbda \quad (28)$$

Equations (26) – (28) can be solved numerically using Mathcad program.

#### 4.3 The E-Bayesian estimation under GELF Loss Function

The Bayes estimator with respect to the GELF loss function is the mean of the posterior distribution as follows:

$$\begin{aligned} \hat{\theta}_{Ge} &= [E_\theta(\theta^{-q})]^{-1/q} = \int_0^\infty \theta^{-1} \pi(\theta|x) d\theta \\ &= \frac{(b+T)^{n+a}}{\Gamma(n+a)} \int_0^\infty \theta^{n+a-q-1} e^{-\theta(b+T)} d\theta = \left[ \frac{\Gamma(n+a-q)}{\Gamma(n+a)(b+T)^{-q}} \right]^{-\frac{1}{q}} \end{aligned} \quad (29)$$

at  $q = 2$

$$\theta_{Ge} = \left[ \frac{\Gamma(n+a-2)}{\Gamma(n+a)(b+T)^{-2}} \right]^{-1/2} = \frac{\sqrt{(n+a-1)(n+a-2)}}{(b+T)} \quad (30)$$

The E – Bayesian estimates  $\hat{\theta}_{EGe1}$ ,  $\hat{\theta}_{EGe2}$  and  $\hat{\theta}_{EGe3}$  based on  $\pi_1(a, b)$ ,  $\pi_2(a, b)$  and  $\pi_3(a, b)$  respectively relative to GELF are the following:

$$1. \hat{\theta}_{EGe1} = \int_0^1 \int_0^c \pi_1(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{\sqrt{(n+a-1)(n+a-2)} \frac{2(c-b)}{c^2}}{(b+T)} dbda \quad (31)$$

$$2. \hat{\theta}_{EGe2} = \int_0^1 \int_0^c \pi_2(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{\sqrt{(n+a-1)(n+a-2)} \frac{1}{c}}{(b+T)} dbda \quad (32)$$

and

$$3. \hat{\theta}_{EGe3} = \int_0^1 \int_0^c \pi_3(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \frac{\sqrt{(n+a-1)(n+a-2)} \frac{2b}{c^2}}{(b+T)} dbda \quad (33)$$

Equations (31) – (33) can be solved numerically using Mathcad program.

#### 4.4 The E-Bayesian estimation under Precautionary Loss Function

The Bayes estimator with respect to the precautionary loss function is the mean of the posterior distribution as follows:

The E – Bayesian estimates  $\hat{\theta}_{EP1}$ ,  $\hat{\theta}_{EP2}$  and  $\hat{\theta}_{EP3}$  based on  $\pi_1(a, b)$ ,  $\pi_2(a, b)$  and  $\pi_3(a, b)$  respectively relative to precautionary loss function are the following:

$$\theta_{Ep} = \frac{E(\theta|X)}{E(\theta^{-1}|X)} = \frac{\frac{(b+T)^{n+a}}{\Gamma(n+a)} \int_0^\infty \theta \theta^{n+a-1} e^{-\theta(b+T)} d\theta}{\frac{(b+T)^{n+a}}{\Gamma(n+a)} \int_0^\infty \frac{1}{\theta} \theta^{n+a-1} e^{-\theta(b+T)} d\theta} = \frac{\sqrt{(n+a-1)(n+a)}}{b+T}$$

$$1. \hat{\theta}_{EP1} = \int_0^1 \int_0^c \pi_1(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left( \frac{\sqrt{(n+a-1)(n+a)}}{b+T} \right) \frac{2(c-b)}{c^2} dbda \quad (34)$$

$$2. \hat{\theta}_{EP2} = \int_0^1 \int_0^c \pi_2(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left( \frac{\sqrt{(n+a-1)(n+a)}}{b+T} \right) \frac{1}{c} dbda \quad (35)$$

and

$$3. \hat{\theta}_{EP3} = \int_0^1 \int_0^c \pi_3(a, b) \hat{\theta}_B dbda = \int_0^1 \int_0^c \left( \frac{\sqrt{(n+a-1)(n+a)}}{b+T} \right) \frac{2b}{c^2} dbda \quad (36)$$

Equations (34) – (36) can be solved numerically using Mathcad program.

#### 5. Hierarchical Bayesian Estimation

In this section, the hierarchical Bayesian estimates for shape parameter of Burr Type II based on SELF, LINX, GELF and PLF are derived.

Based to Lindley and Smith (1972) if  $a, b$  are hyper parameters in  $\theta$  the prior density function of  $\theta$  is  $\pi(\theta|a, b)$  given in Eq. (15) and the hyper prior distributions of  $a, b$  are given in Eqs. (18), (19) and (20) then the corresponding hierarchical prior distributions of  $\theta$  are given in the following:



$$\pi_4(\theta) = \int_0^1 \int_0^c \pi(\theta|a, b) \pi_1(b) db da = \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \frac{2(c-b)}{c^2} db da \quad (37)$$

$$\pi_5(\theta) = \int_0^1 \int_0^c \pi(\theta|a, b) \pi_2(b) db da = \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \frac{1}{c} db da \quad (38)$$

and

$$\pi_6(\theta) = \int_0^1 \int_0^c \pi(\theta|a, b) \pi_3(b) db da = \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \frac{2b}{c^2} db da \quad (39)$$

According to Bayesian theorem, the hierarchical posterior distributions of  $\theta$  can be derived by combining Eqs. (13), (37), (38) and (39) to be

$$\begin{aligned} \pi_1(\theta|a, b) &= \frac{L(\theta|x)\pi_4(\theta)}{\int_0^\infty L(\theta|x)\pi_4(\theta)d\theta} \\ &= \frac{\int_0^1 \int_0^c \left(\frac{2(c-b)}{c^2}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} db da}{\int_0^1 \int_0^c \int_0^\infty \left(\frac{2(c-b)}{c^2}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} \theta^{a-1} d\theta db da} \end{aligned} \quad (40)$$

$$\begin{aligned} \pi_2(\theta|a, b) &= \frac{L(\theta|x)\pi_5(\theta)}{\int_0^\infty L(\theta|x)\pi_5(\theta)d\theta} \\ &= \frac{\int_0^1 \int_0^c \left(\frac{1}{c}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} db da}{\int_0^1 \int_0^c \int_0^\infty \left(\frac{1}{c}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} \theta^{a-1} d\theta db da} \end{aligned} \quad (41)$$

and

$$\begin{aligned} \pi_3(\theta|a, b) &= \frac{L(\theta|x)\pi_6(\theta)}{\int_0^\infty L(\theta|x)\pi_6(\theta)d\theta} \\ &= \frac{\int_0^1 \int_0^c \left(\frac{2b}{c^2}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} \theta^{a-1} db da}{\int_0^1 \int_0^c \int_0^\infty \left(\frac{2b}{c^2}\right) \theta^{n+a-1} e^{-\theta(b+T)} \frac{b^a}{\Gamma(a)} \theta^{a-1} d\theta db da} \end{aligned} \quad (42)$$

### 5.1 Hierarchical Bayesian estimation under SELF

Assuming SELF in Eq. (3) the hierarchical posterior distributions in Eqs. (40), (41) and (42) then the hierarchical Bayes estimates  $\hat{\theta}_{Hse1}$ ,  $\hat{\theta}_{Hse2}$  and  $\hat{\theta}_{Hse3}$  of  $\theta$  are the following:

$$\begin{aligned} \hat{\theta}_{Hse1} &= E[\pi_1(\theta|a, b)] \\ &= \frac{\int_0^\infty \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \theta^{n+a} e^{-\theta(b+T)} \frac{2(c-b)}{c^2} db da d\theta}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2(c-b)}{c^2} db da} \end{aligned} \quad (43)$$

$$\begin{aligned}\hat{\theta}_{Hse2} &= E[\pi_2(\theta|a, b)] \\ &= \frac{\int_0^\infty \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \theta^{n+a} e^{-\theta(b+T)} \frac{1}{c} dbdad\theta}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{1}{c} dbdba}\end{aligned}\quad (44)$$

and

$$\begin{aligned}\hat{\theta}_{Hse3} &= E[\pi_3(\theta|a, b)] \\ &= \frac{\int_0^\infty \int_0^1 \int_0^{1c} \frac{b^a}{\Gamma(a)} \theta^{n+a} e^{-\theta(b+T)} \frac{2b}{c^2} dbdad\theta}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2b}{c^2} dbda}\end{aligned}\quad (45)$$

## 5.2 Hierarchical Bayesian estimation under LINEX

Assuming LINEX in Eq. (5) the hierarchical posterior distributions in Eqs. (40), (41) and (42) then the hierarchical Bayes estimates  $\hat{\theta}_{Hlnx1}$ ,  $\hat{\theta}_{HB2}$  and  $\hat{\theta}_{HB3}$  of  $\theta$  are the following:

$$\begin{aligned}\hat{\theta}_{Hlnx1} &= \frac{-1}{v} \ln E(e^{-v\theta}|X) \\ &= \frac{-1}{v} \frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \frac{2(c-b)}{c^2} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2(c-b)}{c^2} dbda}\end{aligned}\quad (46)$$

$$\begin{aligned}\hat{\theta}_{Hlnx2} &= \frac{-1}{v} \ln E(e^{-v\theta}|X) \\ &= \frac{-1}{v} \frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \frac{1}{c} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{1}{c} dbda}\end{aligned}\quad (47)$$

and

$$\begin{aligned}\hat{\theta}_{Hlnx3} &= \frac{-1}{v} \ln E(e^{-v\theta}|X) \\ &= \frac{-1}{v} \frac{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{(b+T)^{n+a}}{(v+b+T)^{n+a}} \frac{2b}{c^2} dbda}{\int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2b}{c^2} dbda}\end{aligned}\quad (48)$$

## 5.3 Hierarchical Bayesian estimation under GELF

Assuming GELF in Eq. (7) the hierarchical posterior distributions in Eqs. (40), (41) and (42) then the hierarchical Bayes estimates  $\hat{\theta}_{HGe1}$ ,  $\hat{\theta}_{HGe2}$  and  $\hat{\theta}_{HGe3}$  of  $\theta$  are the following:

$$\hat{\theta}_{HGe1} = \frac{\left[ \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-q)}{(b+T)^{n+a-q}} \frac{2(c-b)}{c^2} dbda \right]^{-1/q}}{\left[ \int_0^c \int_0^1 \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2(c-b)}{c^2} dbda \right]^{-1/q}} \quad (49)$$

$$\hat{\theta}_{HGe2} = \frac{\left[ \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-q)}{(b+T)^{n+a-q}} \frac{1}{c} dbda \right]^{-1/q}}{\left[ \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{1}{c} dbda \right]^{-1/q}} \quad (50)$$

$$\hat{\theta}_{HGe3} = \frac{\left[ \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a-q)}{(b+T)^{n+a-q}} \frac{2b}{c^2} dbda \right]^{-1/q}}{\left[ \int_0^1 \int_0^c \frac{b^a}{\Gamma(a)} \frac{\Gamma(n+a)}{(b+T)^{n+a}} \frac{2b}{c^2} dbda \right]^{-1/q}} \quad (51)$$

#### 5.4 Hierarchical Bayesian estimation under PLF

Assuming PLF in Eq. (10) the hierarchical posterior distributions in Eqs. (40), (41) and (42) then the hierarchical Bayes estimates  $\hat{\theta}_{Hp1}$ ,  $\hat{\theta}_{Hp2}$  and  $\hat{\theta}_{Hp3}$  of  $\theta$  are the following:

$$\hat{\theta}_{Hp1} = \sqrt{\frac{\hat{\theta}_{HGe1}}{\hat{\theta}_{Ese1}}} \quad (52)$$

$$\hat{\theta}_{Hp2} = \sqrt{\frac{\hat{\theta}_{HGe2}}{\hat{\theta}_{Ese2}}} \quad (53)$$

$$\hat{\theta}_{Hp3} = \sqrt{\frac{\hat{\theta}_{HGe3}}{\hat{\theta}_{Ese3}}} \quad (54)$$

### 6. Properties of the E – Bayesian and Hierarchical Bayesian Estimates

In this section, we shall discuss the properties of E-Bayesian estimates and the relations among the E – Bayesian and hierarchical Bayesian estimates.

#### 6.1 The relations between the E-Bayesian estimates

In this subsection, we will construct the relations between the E-Bayesian estimates.

##### 6.1.1 Relations among $\hat{\theta}_{Esei}$ ( $i = 1,2,3$ ) from Eqs. (22) – (24)

- (i)  $\hat{\theta}_{Ese3} < \hat{\theta}_{Ese2} < \hat{\theta}_{Ese1}$
- (ii)  $\lim_{H \rightarrow \infty} \hat{\theta}_{Ese1} = \lim_{H \rightarrow \infty} \hat{\theta}_{Ese2} = \lim_{H \rightarrow \infty} \hat{\theta}_{Ese3}$

##### 6.1.2 Relations among $\hat{\theta}_{Elnxi}$ ( $i = 1,2,3$ ) from Eqs. (26) – (28)

- (i)  $\hat{\theta}_{Elnx3} < \hat{\theta}_{Elnx2} < \hat{\theta}_{Elnx1}$

$$(ii) \lim_{H \rightarrow \infty} \hat{\theta}_{Elnx1} = \lim_{H \rightarrow \infty} \hat{\theta}_{Elnx2} = \lim_{H \rightarrow \infty} \hat{\theta}_{Elnx3}$$

6.1.3 Relations among  $\hat{\theta}_{EGei}$  ( $i = 1,2,3$ ) from Eqs. (31) – (33)

$$(i) \hat{\theta}_{EGe3} < \hat{\theta}_{EGe2} < \hat{\theta}_{EGe1}$$

$$(ii) \lim_{H \rightarrow \infty} \hat{\theta}_{EGe1} = \lim_{H \rightarrow \infty} \hat{\theta}_{EGe2} = \lim_{H \rightarrow \infty} \hat{\theta}_{EGe3}$$

6.1.4 Relations among  $\hat{\theta}_{EPi}$  ( $i = 1,2,3$ ) from Eqs. (34) – (36)

$$(i) \hat{\theta}_{EP3} < \hat{\theta}_{EP2} < \hat{\theta}_{EP1}$$

$$(ii) \lim_{H \rightarrow \infty} \hat{\theta}_{EP1} = \lim_{H \rightarrow \infty} \hat{\theta}_{EP2} = \lim_{H \rightarrow \infty} \hat{\theta}_{EP3}$$

## 6.2 The relations between the E-Bayesian and hierarchical Bayesian estimates

In this subsection, we will construct the relations between the E-Bayesian and the hierarchical Bayesian estimates.

6.2.1 Relations among  $\hat{\theta}_{Esei}$  and  $\hat{\theta}_{Hsei}$  ( $i = 1,2,3$ )

It follows from Eqs. (22), (23), (24), (43), (44) and (45) that

$$\lim_{H \rightarrow \infty} \hat{\theta}_{Esei} = \lim_{H \rightarrow \infty} \hat{\theta}_{Hsei} \quad (i = 1,2,3)$$

6.2.2 Relations among  $\hat{\theta}_{Elnxi}$  and  $\hat{\theta}_{Hlnxi}$  ( $i = 1,2,3$ )

It follows from Eqs. (26), (27), (28), (46), (47) and (48) that

$$\lim_{H \rightarrow \infty} \hat{\theta}_{Elnxi} = \lim_{H \rightarrow \infty} \hat{\theta}_{Hlnxi} \quad (i = 1,2,3)$$

6.2.3 Relations among  $\hat{\theta}_{EGei}$  and  $\hat{\theta}_{HGei}$  ( $i = 1,2,3$ )

It follows from Eqs. (30), (31), (32), (49), (50) and (51) that

$$\lim_{H \rightarrow \infty} \hat{\theta}_{EGei} = \lim_{H \rightarrow \infty} \hat{\theta}_{HGei} \quad (i = 1,2,3)$$

6.2.4 Relations among  $\hat{\theta}_{EPi}$  and  $\hat{\theta}_{HPi}$  ( $i = 1,2,3$ )

It follows from Eqs. (34), (35), (36), (52), (53) and (54) that

$$\lim_{H \rightarrow \infty} \hat{\theta}_{EPi} = \lim_{H \rightarrow \infty} \hat{\theta}_{HPi} \quad (i = 1,2,3)$$

## 7. Monte Carlo Simulation

In this section a Monte Carlo Simulation is performed to assess the performance of the E – Bayesian estimates and the hierarchical estimates associated to shape parameter of the Burr Type

II distribution discussed in the previous sections. The simulation structure can be described in the following steps:

1. Set the true value of  $\theta$  at ( $\theta = 0.9$ ). We considered different sample sizes ( $n = 30,50$ ) to study their effect on the resulting estimates.
2. For given values of ( $a = 1, b = 0.5, c = 1.2$ ) we generate  $\theta$  from gamma distribution.
3. Calculate Bayes, E-Bayes, hierarchical Bayes estimates of the unknown shape parameter associated to the Burr Type II distribution according to the formulas that have been obtained.
4. We repeated this process 1000 times and compute the Mean Square Error (MSE) for the estimates where  $MSE(\hat{\theta}) = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}_i - \theta)^2$  where  $\hat{\theta}$  is the estimate of  $\theta$ . The simulation results are displayed in Tables (1 – 4).

Table (1): Bayes, E – Bayesian and hierarchical Bayesian estimates based on (*SELF*) for ( $m = 1000, n = 30,50, \theta = 0.9$ )

n	Bayes		$\theta_{Ese}$		$\theta_{Hse}$	
			estimate	MSE	estimate	MSE
30	Prior 1	0.951	0.939	0.033	0.94	0.034
	Prior 2		0.933	0.032	0.935	0.033
	Prior 3		0.927	0.031	0.931	0.032
50	Prior 1	0.931	0.924	0.019	0.865	0.022
	Prior 2		0.921	0.018	0.863	0.022
	Prior 3		0.917	0.018	0.861	0.022

Table (2): Bayes, E – Bayesian and hierarchical Bayesian estimates based on LINEX loss function for ( $m = 1000, n = 30,50, \theta = 0.9$ )

n	Bayes		$\theta_{Elnx}$		$\theta_{Hlnx}$	
			estimate	MSE	estimate	MSE
30	Prior 1	0.882	0.871	0.024	0.169	0.538
	Prior 2		0.866	0.024	0.16	0.033
	Prior 3		0.861	0.024	0.154	0.56
50	Prior 1	0.89	0.883	0.015	0.092	0.654
	Prior 2		0.88	0.015	0.087	0.022
	Prior 3		0.877	0.015	0.084	0.667

Table (3): Bayes, E – Bayesian and hierarchical Bayesian estimates based on (*GELF*) for ( $m = 1000, n = 30, 50, \theta = 0.9$ )

$n$	Bayes		$\theta_{EGe}$		$\theta_{HGe}$	
			estimate	MSE	estimate	MSE
30	Prior 1	0.905	0.893	0.029	0.949	0.01
	Prior 2		0.887	0.028	0.947	0.0099
	Prior 3		0.881	0.028	0.945	0.0097
50	Prior 1	0.903	0.896	0.017	0.918	0.0058
	Prior 2		0.893	0.017	0.917	0.0057
	Prior 3		0.89	0.016	0.916	0.0056

Table (4): Bayes, E – Bayesian and hierarchical Bayesian estimates based on (*PLF*) for ( $m = 1000, n = 30, 50, \theta = 0.9$ )

$n$	Bayes		$\theta_{EGP}$		$\theta_{HGP}$	
			estimate	MSE	estimate	MSE
30	Prior 1	0.966	0.954	0.036	0.99	0.01
	Prior 2		0.948	0.034	0.947	0.006
	Prior 3		1.131	0.081	0.987	0.0097
50	Prior 1	0.94	0.933	0.019	0.967	0.0059
	Prior 2		0.93	0.019	0.917	0.0042
	Prior 3		0.926	0.017	0.966	0.0058

**Conclusions:**

1. MSE of E-Bayesian estimates under LINEX loss function are less than E-Bayesian estimates under other error loss functions
2. MSE of Bayesian and E-Bayesian estimates decrease as  $n, m$  increases

**8. Application**

Data of 56 patients with hyperer nephroma is used to apply Burr Type II distribution on it. This data is displayed in Table (5) as follows.

Table (5): Real data

0	4	-9	2	2	0	0	0	-9	0
1.5	5	4	0	0	4	0	2	1	1.5
0	0	-9	8	-9	0	2	10	2	12
-9	5	0	-9	0	0	0	5	8	8
-9	0	1	-9	2	1	-9	0	3	0
0.5	3	0	2	2	8				

Using the one-sample KS test for goodness of fit, the  $p - value = 1.63917 \times 10^{-7}$  which is less than the significance level  $\alpha = 0.01$  so the null hypothesis is rejected so this real data has the Burr Type II distribution.

The Bayes, E – Bayesian and the hierarchical Bayesian estimate for the parameter  $\theta$  under the four loss functions (SELF, LINEX, GELF and PELF) are obtained in Table (6).

Table (6): Bayes, E – Bayesian and hierarchical Bayesian estimates for the parameter  $\theta$ 

Loss function	Bayes	E - Bayesian			Hierarchical		
		Prior 1	Prior 2	Prior 3	Prior 1	Prior 2	Prior 3
SELF	0.577	0.573	0.572	0.571	0.573	0.572	0.571
LINEX	0.563	0.559	0.557	0.557	0.046	0.043	0.04
GELF	0.562	0.557	0.556	0.556	0.75	0.75	0.749
PELF	0.583	0.578	0.577	0.577	0.874	0.874	0.873

### Conclusions:

1. It is noticed that the Bayes estimates are less than the ones obtained from the simulation.
2. Also the E – Bayesian estimates are less than the ones obtained from the simulation.
3. In addition the hierarchical estimates are less than the ones obtained from the simulation.

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